NASA TECHNICAL TRANSLATION NASA TT F-11,618

NASA TT F-11,618

EMISSION OF PHOTONS AND ELECTRON-POSITRON PAIRS IN A MAGNETIC FIELD

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Translation of "Izlucheniye Fotonov i Elektronno-Pozitronnykh Par v Magnitnom Pole"
Zhurnal Eksperimental'noy i Teoreticheskoy Fiziki,
Vol. 26, No. 1, pp. 19-34, 1954

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 MARCH 1968

EMISSION OF PHOTONS AND ELECTRON-POSITRON PAIRS IN A MAGNETIC FIELD

N. P. Klepikov

ABSTRACT. This article is a study of the luminous electron, electron-positron pair production, and single-photon annihilation of pairs in a strong magnetic field from the standpoint of relativistic quantum theory. Stationary wave functions are derived and the probability of particle transformations in a magnetic field is examined. Asymptotic equations are derived. The emission spectrum of the luminous electron is examined with consideration of the electromagnetic field as a perturbation. The probabilities of single-photon and two-photon annihilation of electron-positron pairs are compared. Transformations of a photon to an electron-positron pair and of electrons to electron-photon pairs are discussed.

The phenomenon of the luminous electron, pair production by photons and electrons, and the single-photon annihilation of pairs in a magnetic field are examined using relativistic quantum theory. The results obtained are valid for very strong magnetic fields, where the emission intensity of electrons differs substantially from the intensity calculated by classical theory. /19*

1. Wave Functions and Matrix Elements

Quantum theory of the luminous electron has recently drawn the attention of many investigators who have obtained, however, only partial results. The need arises, therefore, to examine the theory of this phenomenon in greater detail than it has been studied up to now, and also to construct the theory of other particle transformations in a magnetic field with the help of this mathematical tool.

This article is the development of articles [1-4] on quantum theory of the luminous electron, where the applicability of computations of emission intensity is not limited here to the trivial consideration of quantum corrections to classical formulas. Also examined are phenomena related to the creation and annihilation of electron-positron pairs in a magnetic field.

In determining the probability of particle transformations in a magnetic field we will, first of all, find the stationary wave functions of electrons in the magnetic field as eigenfunctions of the following complete system of

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* Numbers in the margin indicate pagination in the foreign text.

commuting operators: energy operator

$$\hat{\mathcal{H}} = c\rho_1\vec{\sigma}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right) + \rho_3 mc^2, \tag{1.1}$$

operator of the projection of momentum on the magnetic field

(1.2)

operator of the projection of momentum on the spin

$$\hat{p}_\sigma = \vec{\sigma}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right) \tag{1.3}$$

and operator of the y-coordinate of the center of rotation

$$\hat{y}_0 = -\frac{1}{i}\frac{c\hbar}{eH}\frac{\partial}{\partial x}, \tag{1.4}$$

assuming that the z axis is directed along the magnetic field and the vector potential has the form

$$A_x = -Hy, \; A_y = A_z = 0. \tag{1.5}$$

The eigenfunctions of this system of operators are as follows: /20

$$\psi = \frac{1}{\sqrt{L_x L_z}} \exp\left\{-ic\varepsilon Kt + ik_1 x + ik_3 z - \frac{\xi^2}{2}\right\} \begin{Bmatrix} \alpha & A & \overline{\text{II}}_l(\xi) \\ s\alpha & B & \overline{\text{II}}_{l-1}(\xi) \\ s\varepsilon\beta & A & \overline{\text{II}}_l(\xi) \\ \varepsilon\beta & B & \overline{\text{II}}_{l-1}(\xi) \end{Bmatrix}, \tag{1.6}$$

$$\alpha = \sqrt{\frac{1}{2}\left(1 + \varepsilon\frac{k_0}{K}\right)}, \quad \beta = \sqrt{\frac{1}{2}\left(1 - \varepsilon\frac{k_0}{K}\right)}, \tag{1.7}$$

$$A = \sqrt{\frac{1}{2}\left(1 + s\frac{k_3}{k}\right)}, \quad B = \sqrt{\frac{1}{2}\left(1 - s\frac{k_3}{k}\right)}, \tag{1.8}$$

$\overline{H}_l(\xi) = \sqrt{\frac{1}{2^l l!}\sqrt{\frac{\gamma}{\pi}}}\ H_l\ (\xi)$ is the standard Hermite polynomial,

$\xi = y\sqrt{\gamma} + \frac{k_1}{\sqrt{\gamma}}$, $k = \sqrt{k_0^2 + 2l\gamma}$, $sk\hbar$ is the eigenvalue of operator (1.3), $s = \pm 1$, $l = 0, 1, 2, \ldots$ is an integer, $\gamma = eH/c\hbar$. The energy of the electron $E = \varepsilon c\hbar K$, $\varepsilon = \pm 1$,

$$K = \sqrt{k_0^2 + k_3^2 + 2l\gamma}, \tag{1.9}$$

momentum along the field $p_z = \hbar k_3$, $-\infty < k_3 < \infty$ and the y-coordinate of the center of rotation $y_0 = -k_1/\gamma$, $-\infty < k_1 < \infty$. From (1.9) follows

$$l = \frac{H_0}{2H}\left[\left(\frac{E}{mc^2}\right)^2 - 1 - \left(\frac{p_z}{mc}\right)^2\right], \tag{1.10}$$

where $H_0 = m^2c^3/e\hbar = 4.67 \cdot 10^{13}$Oe. By transforming the coordinates and calibration of the potential it is easy to establish the correspondence of the functions of (1.6) to the functions found by other authors for other sets of operators (see [3]).

We will now examine the matrix elements of the operator for the respective wave functions of (1.6). By integrating with respect to the x and z coordinates we find $\delta_{k_1', \varkappa_1 + k_1}\delta_{k_3', \varkappa_3 + k_3}$, and the integral with respect to y yields

$$\begin{aligned}\overline{\alpha}_x = \alpha' A' \varepsilon\beta B\, I(l', l-1) + s'\alpha' B' \varepsilon s\beta A\, I(l'-1, l) + \\ + \varepsilon' s'\beta' A' s\alpha B\, I(l', l-1) + \varepsilon'\beta' B'\alpha A\, I(l'-1, l),\end{aligned} \tag{1.11}$$

$$\begin{aligned}\overline{\alpha}_y = i\{-\alpha' A'\varepsilon\beta B\, I(l', l-1) + s'\alpha' B'\varepsilon s\beta A\, I(l'-1, l) - \\ - \varepsilon' s'\beta' A' s\alpha B\, I(l', l-1) + \varepsilon'\beta' B'\alpha A\, I(l'-1, l)\},\end{aligned} \tag{1.12}$$

$$\begin{aligned}\overline{\alpha}_z = \alpha' A'\varepsilon s\beta A\, I(l', l) - s'\alpha' B'\varepsilon\beta B\, I(l'-1, l-1) + \\ + \varepsilon' s'\beta' A'\alpha A\, I(l', l) - \varepsilon'\beta' B' s\alpha B\, I(l'-1, l-1),\end{aligned} \tag{1.13}$$

where the coefficients are determined by equations (1.7) and (1.8) and

$$I(l', l) = \frac{\sqrt{\gamma}}{\sqrt{l!\, l'!\, 2^{l+l'} \pi}} \int_{-\infty}^{\infty} \mathrm{H}_{l'}\left(y\sqrt{\gamma} + \frac{k_1 + \varkappa_1}{\sqrt{\gamma}}\right) \mathrm{H}_l\left(y\sqrt{\gamma} + \frac{k_1}{\sqrt{\gamma}}\right) \times$$
$$\times \exp\left\{-\frac{1}{2}\left(y\sqrt{\gamma} + \frac{k_1 + \varkappa_1}{\sqrt{\gamma}}\right)^2 - \frac{1}{2}\left(y\sqrt{\gamma} + \frac{k_1}{\sqrt{\gamma}}\right)^2 + i\varkappa_2 y\right\} dy =$$
$$= \frac{1}{(l-l')!}\sqrt{\frac{l!}{l'!}} \exp\left\{-\frac{q}{2} - i(l-l')\varphi - i\varkappa_2 \frac{2k_1 + \varkappa_1}{2\gamma}\right\} \times$$
$$\times q^{(l-l')/2}\, {}_1F_1(-l'; l-l'+1; q), \tag{1.14}$$

where $l \geq l'$, $q = (\varkappa_1^2 + \varkappa_2^2)/2\gamma$, $\varphi = \operatorname{arctg}(\varkappa_2/\varkappa_1)$. Integral (1.14) is found by converting to the variable $z = y\sqrt{\gamma} + (2k_1 + \varkappa_1 - i\varkappa_2)/2\sqrt{\gamma}$ [the integration curve for the straight line $z = -(i\varkappa_2/2\sqrt{\gamma}) + x$], transferring the curve of integration to the true axis, using the equality /21

$$\mathrm{H}_l(z + a) = \sum_{k=0}^{l} \frac{l!}{k!\,(l-k)!}(2a)^k \mathrm{H}_{l-k}(z) \tag{1.15}$$

and making use of the orthogonality of Hermite's polynomials. By using the relation

$${}_1F_1(-l'; l-l'+1; z) = \frac{(l-l')!}{l!}(-z)^{l'}\, {}_2F_0\left(-l', -l; -\frac{1}{z}\right) \tag{1.16}$$

it is easy to find (considering the symmetry of the function of ${}_2F_0$ with respect to its parameters), that

$$I(l', l)\, e^{il(2\varphi - \pi)} = I(l, l')\, e^{il'(2\varphi - \pi)}. \tag{1.17}$$

Summation of the spin variables s and s' yields

$$\sum_{s,\, s'=\pm 1} |\bar{\alpha}_x|^2 = \frac{KK' - \varepsilon\varepsilon' k_0^2 - \varepsilon\varepsilon' k_3 k_3'}{2KK'}\left[|I(l', l-1)|^2 + |I(l'-1, l)|^2\right] +$$
$$+ \frac{\varepsilon\varepsilon'\gamma\sqrt{ll'}}{KK'}\left[I^+(l', l-1)\, I(l'-1, l) + I^+(l'-1, l)\, I(l', l-1)\right], \tag{1.18}$$

$$\sum_{s,\, s'=\pm 1} |\bar{\alpha}_y|^2 = \frac{KK' - \varepsilon\varepsilon' k_0^2 - \varepsilon\varepsilon' k_3 k_3'}{2KK'} [|I(l',\ l-1)|^2 + |I(l'-1,\ l)|^2] -$$
$$- \frac{\varepsilon\varepsilon'\gamma\sqrt{ll'}}{KK'} [I^+(l',\ l-1)\, I(l'-1,\ l) + I^+(l'-1,\ l)\, I(l',\ l-1)], \tag{1.19}$$

$$\sum_{s,\, s'=\pm 1} |\bar{\alpha}_z|^2 = \frac{KK' - \varepsilon\varepsilon' k_0^2 + \varepsilon\varepsilon' k_3 k_3'}{2KK'} [|I(l',\ l)|^2 + |I(l'-1,\ l-1)|^2] -$$
$$- \frac{\varepsilon\varepsilon'\gamma\sqrt{ll'}}{KK'} [I^+(l',\ l)\, I(l'-1,\ l-1) + I^+(l'-1,\ l-1)\, I(l',\ l)], \tag{1.20}$$

$$\sum_{s,\, s'=\pm 1} (\bar{\alpha}_x^+ \bar{\alpha}_z + \bar{\alpha}_z^+ \bar{\alpha}_x) =$$
$$= \frac{\varepsilon\varepsilon'}{2KK'} \{k_3' \sqrt{2l\gamma}\, [I^+(l',\ l)\, I(l',\ l-1) + I^+(l',\ l-1)\, I(l',\ l) +$$
$$+ I^+(l'-1,\ l-1)\, I(l'-1,\ l) + I^+(l'-1,\ l)\, I(l'-1,\ l-1)] +$$
$$+ k_3 \sqrt{2l'\gamma}\, [I^+(l'-1,\ l-1)\, I(l',\ l-1) + I^+(l',\ l-1)\, I(l'-1,\ l-1) +$$
$$+ I^+(l',\ l)\, I(l'-1,\ l) + I^+(l'-1,\ l)\, I(l',\ l)]\}, \tag{1.21}$$

$$\sum_{s,\, s'=\pm 1} (\bar{\alpha}_y^+ \bar{\alpha}_z + \bar{\alpha}_z^+ \bar{\alpha}_y) =$$
$$= \frac{i\varepsilon\varepsilon'}{2KK'} \{k_3' \sqrt{2l\gamma}\, [-I^+(l',\ l)\, I(l',\ l-1) + I^+(l',\ l-1)\, I(l',\ l) +$$
$$+ I^+(l'-1,\ l-1)\, I(l'-1,\ l) - I^+(l'-1,\ l)\, I(l'-1,\ l-1)] +$$
$$+ k_3 \sqrt{2l'\gamma}\, [-I^+(l'-1,\ l-1)\, I(l',\ l-1) + I^+(l',\ l-1)\, I(l'-1,\ l-1) +$$
$$+ I^+(l',\ l)\, I(l'-1,\ l) - I^+(l'-1,\ l)\, I(l',\ l)]\}, \tag{1.22}$$

$$\sum_{s,\, s'=\pm 1} (\bar{\alpha}_x^+ \bar{\alpha}_y + \bar{\alpha}_y^+ \bar{\alpha}_x) =$$
$$= 2i \frac{\varepsilon\varepsilon'\gamma\sqrt{ll'}}{KK'} [I^+(l',\ l-1)\, I(l'-1,\ l) - I^+(l'-1,\ l)\, I(l',\ l-1)]. \tag{1.23}$$

Henceforth it will be necessary also to represent the integrals of (1.14) /22
in another form. By using the relations

$$z\, {}_1F_1(\alpha+1;\ \gamma+1;\ z) = \gamma\, {}_1F_1(\alpha+1;\ \gamma;\ z) - \gamma\, {}_1F_1(\alpha;\ \gamma;\ z), \tag{1.24}$$

$$\alpha\, {}_1F_1(\alpha+1;\ \gamma+1;\ z) = (\alpha-\gamma)\, {}_1F_1(\alpha;\ \gamma+1;\ z) + \gamma\, {}_1F_1(\alpha;\ \gamma;\ z) \tag{1.25}$$

it is easy to obtain

$$e^{-i\varphi}\sqrt{q}\, I(l',\ l-1) = -\sqrt{l'}\, I(l'-1,\ l-1) + \sqrt{l}\, I(l',\ l), \tag{1.26}$$

$$e^{i\varphi}\sqrt{q}\, I(l'-1,\ l) = \sqrt{l}\, I(l'-1,\ l-1) - \sqrt{l'}\, I(l',\ l). \tag{1.27}$$

Then, by using the equation

$$\frac{d}{dz}\, {}_1F_1(\alpha;\ \gamma;\ z) = \frac{\alpha}{\gamma}\, {}_1F_1(\alpha+1;\ \gamma+1;\ z) \tag{1.28}$$

we find

$$\sqrt{ll'}\, I(l'-1,\ l-1) = \frac{1}{2}(l+l'-q)\, I(l',l) - q\frac{\partial I(l'\ l)}{\partial q}. \tag{1.29}$$

Combining (1.29) with (1.26) and (1.27) we have

$$e^{-i\varphi}\sqrt{lq}\, I(l',\ l-1) = \frac{1}{2}(q+l-l')\, I(l',\ l) + q\partial I(l',\ l)/\partial q, \tag{1.30}$$

$$e^{i\varphi}\sqrt{l'q}\, I(l'-1,l) = -\frac{1}{2}(q+l'-l)\, I(l',\ l) - q\partial I(l',\ l)/\partial q. \tag{1.31}$$

Thus, the four integrals $I(l',\ l)$, $I(l',\ l-1)$, $I(l'-1,\ l)$, $I(l'-1,\ l-1)$ are expressed through the two functions $I(l',\ l)$ and $\partial I(l',\ l)/\partial q$.

2. Asymptotic Formulas For Matrix Elements With Large Parameters

In order to avail ourselves of the possibility of investigating particle transformations for large energies, we will have to find the asymptotic formulas for the integrals of (1.14) entering into the matrix elements, which are useful in those cases where the numbers l', $l - l'$ and q are simultaneously large. After substituting $-l' \to -l\cos^2\alpha$, $l-l'+1 \to l\sin^2\alpha + 1$ $\left(0 \leqslant \alpha < \frac{\pi}{2}\right)$ and $q \to l\zeta$ in the parameters of the degenerate hypergeometric function ${}_1F_1(-l';\ l-l'+1;\ q)$, we will make use of the usual integral representation

$${}_1F_1(-l\cos^2\alpha;\ l\sin^2\alpha+1;\ l\zeta)=$$

$$=\frac{1}{2\pi i}\frac{\Gamma(l\cos^2\alpha+1)\,\Gamma(l\sin^2\alpha+1)}{\Gamma(l+1)}\int\limits_C \exp\{l[\zeta t+\ln(1-t)-\cos^2\alpha(\ln t+\pi i)]\}\frac{dt}{t}, \qquad (2.1)$$

where the contour C passes around the point $t = 0$ in the positive direction. The expression after the exponent sign in the integral of (2.1) has stationary points under the relation

$$t_0=\left[\zeta-\sin^2\alpha\pm\sqrt{(\zeta-\sin^2\alpha)^2-4\zeta\cos^2\alpha}\right]/2\zeta. \qquad (2.2)$$

The points t_0 lie on the real axis if

$$\zeta\leqslant(1-\cos\alpha)^2,\quad \zeta\geqslant(1+\cos\alpha)^2, \qquad (2.3)$$

and, consequently, the function under consideration will be monotonic for $l \to \infty$. Otherwise, the points t_0 are complex, and the asymptotic behavior of the function will have an oscillating character (see [5], pp. 246). We will /23
not examine the latter possibility, since, owing to the law of conservation of energy, the first condition of (2.3) is always satisfied for the luminous electron, and the second condition of (2.3) is satisfied in the case of the creation and annihilation of pairs. Let us assume that

$\zeta = (1-\cos\alpha)^2\operatorname{sech}\beta$. If the second condition of (2.3) is satisfied, all calculations are analogous to the case under consideration, and it will be necessary only to replace $\cos\alpha$ by $-\cos\alpha$ and $\operatorname{sech}\beta$ by $\sec\beta$ $(0\leqslant\beta<\pi/2)$.

The condition $\beta = 0$ gives the boundary of monotonicity of the function of ${}_1F_1$. In the coordinates $l'\geqslant 0,\ n=l-l'\geqslant 0,\ q\geqslant 0$ the equation of the boundary (Figure 1) will have the form

$$n=q\pm 2\sqrt{l'q} \qquad (2.4)$$

(equation of an elliptical cone). The

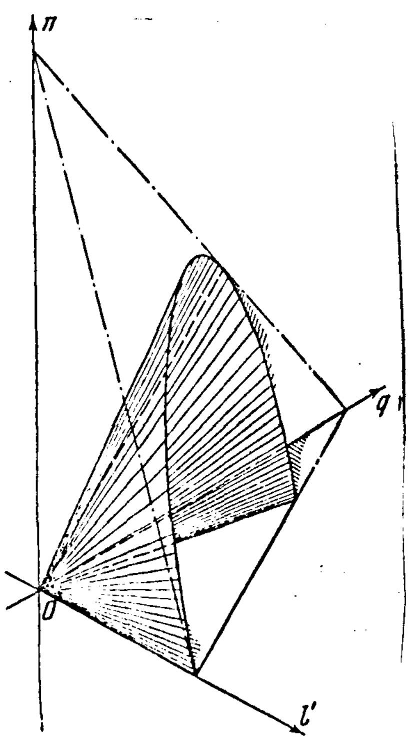

Fig. 1. Boundary of Monotonicity of the Function ${}_1F_1$ $(-l', n+1, q)$

region of monotonicity is located outside the cavity of this cone.

Let us take in (2.2) the positive radical sign [for the second condition of (2.3) it is negative], since it is only here that the integral, after integration, approaches the contour of the steepest descent passing through t_0. We will further substitute $t \to t_0 e^t$ and obtain the equation

$$ {}_1F_1(-l\cos^2\alpha; l\sin^2\alpha+1; l(1-\cos\alpha)^2 \operatorname{sech}\beta) = \frac{1}{2\pi i} \frac{\Gamma(l\cos^2\alpha+1)\Gamma(l\sin^2\alpha+1)}{\Gamma(l+1)} \times $$
$$ \times \exp\{l[(1-\cos\alpha)^2 \operatorname{sech}\beta\, t_0 + \ln(1-t_0) - \cos^2\alpha(\ln t_0 + \pi i)]\} \int_{C'} e^{-l\tau} dt, \tag{2.5} $$

where

$$ -\tau = (1-\cos\alpha)^2 \operatorname{sech}\beta\, t_0(e^t - 1) + \ln\frac{1-t_0 e^t}{1-t_0} - t\cos^2\alpha. \tag{2.6} $$

We will expand $-\tau$ in powers of t:

$$ -\tau = \mu t^2 + \nu t^3 + \sigma t^4 + \ldots, \tag{2.7} $$

where

$$ \mu = {}^1/_2(\cos^2\alpha - B^2), \quad \nu = {}^1/_6(\cos^2\alpha - 3B^2 - 2B^3), $$
$$ \sigma = {}^1/_{24}(\cos^2\alpha - 7B^2 - 12B^3 - 6B^4), \quad B \equiv (1-\cos\alpha)^2 \operatorname{sech}\beta\, t_0 - \cos^2\alpha. \tag{2.8} $$

In particular, for $\operatorname{sech}\beta = 1-\xi$, $\xi \ll 1$,

$$ \mu \approx \sqrt{\xi}\cos^{5/2}\alpha(1-\cos\alpha), \quad \nu \approx -{}^1/_3\cos^2\alpha(1-\cos\alpha), $$
$$ \sigma \approx -{}^1/_4\cos^2\alpha(1-\cos\alpha)^2. \tag{2.9} $$

The conversion of μ to zero on the boundary of monotonicity makes it impossible to obtain for ${}_1F_1$ the asymptotic forumla uniformly suitable for $\xi \to 0$ using the saddle-point method. Therefore, for $l \gg 1$, we will consider not one term in (2.7), but both first terms simultaneously [6]. Then, in addition to the integral with $-\tau$ from (2.7) for the contour $\operatorname{Im}\tau = 0$, we will have the integral with /24

$$ -\tau = \mu T^2 + \nu T^3 \tag{2.10} $$

in the complex plane $T = U + iV$ for the contour

$$V = \pm\sqrt{(2\mu/\nu)\, U + 3U^2}. \tag{2.11}$$

It is easy to see that the difference of these integrals is

$$2i\int_0^\infty e^{-l\tau} \frac{d}{d\tau} \operatorname{Im}(t - T)\, d\tau. \tag{2.12}$$

In (2.12) the values t and T on the upper half of the contour are to be considered. Finding the integral with respect to the T-contour by transferring the curve of integration of the asymptote, we find

$$\frac{1}{2\pi i}\int_{C_T} e^{-l\tau} dT = \frac{\mu}{3\pi\sqrt{3}\,|\nu|} \exp\left\{\frac{2l}{27}\frac{\mu^3}{\nu^2}\right\} K_{1/3}\left(\frac{2l}{27}\frac{\mu^3}{\nu^2}\right), \tag{2.13}$$

where $K_{1/3}$ is the Bessel function of the imaginary argument. Using (2.12) as shown in [3], we may evaluate the upper bound of error in this calculation and finally obtain

$$\begin{aligned} &{}_1F_1(-l\cos^2\alpha;\ l\sin^2\alpha + 1;\ l(1-\cos\alpha)^2 \operatorname{sech}\beta) = \\ &= \frac{\Gamma(l\cos^2\alpha + 1)\,\Gamma(l\sin^2\alpha + 1)}{\Gamma(l+1)} \times \\ &\times \exp\{l[(1-\cos\alpha)^2 \operatorname{sech}\beta\, t_0 + \ln(1-t_0) - \cos^2\alpha\,(\ln t_0 + \pi i)]\} \times \\ &\times \left\{\frac{\mu}{3\pi\sqrt{3}\,|\nu|} \exp\left\{\frac{2l}{27}\frac{\mu^3}{\nu^2}\right\} K_{1/3}\left\{\frac{2l}{27}\frac{\mu^3}{\nu^2}\right\} + \Theta\,\frac{3^{1/6}\,\Gamma(2/3)}{2^{2/3}\pi\,(l\cos^2\alpha)^{2/3}}\right\},\quad |\Theta| \leqslant 1. \end{aligned} \tag{2.14}$$

Now it is easy to see that (2.14) is valid for $l' \gg 1$ and for any sech β. For $\beta \to 0$, we obtain from (2.14) the first two terms of the expansion of ${}_1F_1$ by the saddle-point method. For $\beta \to \infty$ the left half of (2.14) trends toward unity. By substituting in (2.14) cos α by - cos α and β by $i\beta$, the expression obtained, for $\beta \to \frac{\pi}{2}$, trends toward the right half of (1.16), if in the latter formula we replaced the function of ${}_2F_0$ by unity (for $z \gg l$ and $z \gg l'$).

We will now find the approximate expression for the integral $I\ (l', l)$ entering into the matrix elements, assuming that sech $\beta = 1 - \xi$, $\xi \ll 1$. Collecting all factors and expanding both the argument of the function of

$K_{1/3}$ and the product on the left of this function in powers of $\sqrt{\xi}$, we find

$$|I(l\cos^2\alpha, l)| \approx \frac{1}{\pi\sqrt{3}}\sqrt{\xi}\,K_{1/3}\left\{\frac{2l}{3}(1-\cos\alpha)\sqrt{\xi^3\cos\alpha}\right\}, \qquad (2.15)$$

if we disregard the values of the order of $l\xi^2$ in comparison with $l\xi^{1/2}$*. In the original variables

$$|I(l', l)| = \frac{1}{\pi\sqrt{3}}\frac{\sqrt{(\sqrt{l}-\sqrt{l'})^2-q}}{\sqrt{l}-\sqrt{l'}}K_{1/3}\left\{\frac{2}{3}\frac{\sqrt[4]{ll'}\,[(\sqrt{l}-\sqrt{l'})^2-q]^{3/2}}{(\sqrt{l}-\sqrt{l'})^2}\right\}. \qquad (2.16)$$

Differentiating with respect to q, we find /25

$$\left|\frac{\partial I(l', l)}{\partial q}\right| = \frac{1}{\pi\sqrt{3}}\frac{\sqrt[4]{ll'}\,[(\sqrt{l}-\sqrt{l'})^2-q]}{(\sqrt{l}-\sqrt{l'})^3}K_{2/3}\left\{\frac{2}{3}\frac{\sqrt[4]{ll'}\,[(\sqrt{l}-\sqrt{l'})^2-q]^{3/2}}{(\sqrt{l}-\sqrt{l'})^2}\right\}. \qquad (2.17)$$

For the second region of monotonicity

$$|I(l', l)| = \frac{1}{\pi\sqrt{3}}\frac{\sqrt{q-(\sqrt{l}+\sqrt{l'})^2}}{\sqrt{l}+\sqrt{l'}}K_{1/3}\left\{\frac{2}{3}\frac{\sqrt[4]{ll'}\,[q-(\sqrt{l}+\sqrt{l'})^2]^{3/2}}{(\sqrt{l}+\sqrt{l'})^2}\right\}, \qquad (2.18)$$

$$\left|\frac{\partial I(l', l)}{\partial q}\right| = \frac{1}{\pi\sqrt{3}}\frac{\sqrt[4]{ll'}\,[q-(\sqrt{l}+\sqrt{l'})^2]}{(\sqrt{l}+\sqrt{l'})^3}K_{2/3}\left\{\frac{2}{3}\frac{\sqrt[4]{ll'}\,[q-(\sqrt{l}+\sqrt{l'})^2]^{3/2}}{(\sqrt{l}+\sqrt{l'})^2}\right\}. \qquad (2.19)$$

3. Emission Spectrum of the Luminous Electron

We will examine the probability of transition of an electron located in a constant and uniform magnetic field to a lower level of positive energy with the emission of a photon by viewing the electromagnetic field of emission as a perturbation. We will take the energy of perturbation in the usual form:

$$U^+ = \frac{e}{\sqrt{L_x L_y L_z}}\sqrt{\frac{2\pi c\hbar}{\varkappa}}\,\vec{\alpha}\,\mathbf{a}^+(\vec{\varkappa})e^{ic\varkappa t - i\vec{\varkappa}\mathbf{r}}, \qquad (3.1)$$

where $\hbar\vec{\kappa}$ and $c\hbar\vec{\kappa}$ are the momentum and energy of the photon, whereupon

$$a_\lambda(\vec{\varkappa})\,a^+_{\lambda'}(\vec{\varkappa}) = \delta_{\lambda\lambda'} - (\varkappa_\lambda\varkappa_{\lambda'})/\varkappa^2. \qquad (3.2)$$

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Employing the usual calculation according to the method of nonstationary theory of perturbation and assuming that $k_1 = k_3 = 0$, we will find the probability of transition

$$w = \frac{4\pi^2 e^2}{L_x L_y L_z \hbar \varkappa} |\vec{\alpha} a^+|^2 \delta(K - K' - \varkappa)\, \delta_{k_1',\, -\varkappa_1} \delta_{k_3',\, -\varkappa_3}, \tag{3.3}$$

where the vector components of $\vec{\alpha}$ are defined in (1.11)--(1.13). The laws of conservation take into account the recoil of the electron and the conservation of energy ($\varepsilon\varepsilon' = 1$):

$$k_3' = -\varkappa \cos \theta, \tag{3.4}$$

$$k_1' = -\varkappa \sin \theta \cos \varphi, \tag{3.5}$$

$$\sqrt{k_0^2 + 2l\gamma} - \sqrt{k_0^2 + 2l'\gamma + \varkappa^2 \cos^2 \theta} = \varkappa. \tag{3.6}$$

Now it is easy to find

$$\delta(K - K' - \varkappa) = \frac{\sqrt{l' + p + r'}}{\sqrt{l + p} - \sin^2\theta \sqrt{q}}\, \delta\left\{\varkappa - \frac{\sqrt{2(l + p)\gamma}}{\sin^2\theta}\left[1 - \sqrt{1 - \frac{n}{l + p}\sin^2\theta}\right]\right\}, \tag{3.7}$$

where

$$p = \frac{H_0}{2H}, \quad r' = \frac{H_0}{2H}\left(\frac{p_z'}{mc}\right)^2 = \frac{\varkappa^2 \cos^2\theta}{2\gamma}, \quad q = \frac{\varkappa^2 \sin^2\theta}{2\gamma}.$$

By multiplying the probability of emission by the energy of the photons $c\hbar\varkappa$, averaging the initial spin states, and adding the final states, we will find the emission intensity

$$\begin{aligned} I = \frac{ce^2}{4\pi^2} \sum_{n=0}^{l} \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi\, \frac{2\gamma q \sqrt{l' + p + r'}}{\sqrt{l + p} - \sin^2\theta \sqrt{q}} \sum_{s, s' = \pm 1} \{|\overline{\alpha}_x|^2 + |\overline{\alpha}_y|^2 \cos^2\theta + \\ + |\overline{\alpha}_z|^2 \sin^2\theta - \sin\theta\cos\theta\, [\cos\varphi(\overline{\alpha}_x^+ \overline{\alpha}_z + \overline{\alpha}_z^+ \overline{\alpha}_x) + \sin\varphi\, (\overline{\alpha}_y^+ \overline{\alpha}_z + \overline{\alpha}_z^+ \overline{\alpha}_y)] - \\ - \sin^2\theta \sin\varphi\cos\varphi\, (\overline{\alpha}_x^+ \overline{\alpha}_y + \overline{\alpha}_y^+ \overline{\alpha}_x)\}. \end{aligned} \tag{3.8}$$ /26

The expression within the braces in (3.8) is obtained when applying (3.2) to the probability of (3.3), with consideration of the condition that $k_3 = 0$. This expression, as we may easily prove, does not depend on angle ϕ. Therefore, in preserving our notation, we may set $\phi = 0$ in (3.8).

We will now use equations (1.18)--(1.23) and (1.29)--(1.31), assuming that $l \gg 1$ and $l' \gg 1$, and we will consider that $1-[q/(\sqrt{l}-\sqrt{l'})^2] \ll 1$, where the particles have high energies. Then, assuming $\theta = (\pi/2)+\psi$, $\psi \ll \pi/2$, and using (2.16) and (2.17), disregarding the values p and r' in comparison with l and l' in all places where the main terms are not cancelled, and then proceeding from the summation with respect to n to integration, we obtain[1]

$$I = \frac{ce^2}{R}\frac{E}{\hbar c}\frac{1}{3\pi^2}\int_0^l dn \int_{-\pi/2}^{\pi/2} d\psi \, \frac{(\sqrt{l}-\sqrt{l-n})^2 (p+l\psi^2)}{l^{1/2}(l-n)^{3/2}} \times$$
$$\times \Big\{ (2l-n)(p+l\psi^2) K^2_{2/3}\left[\frac{2}{3}\frac{\sqrt{l}-\sqrt{l-n}}{\sqrt{l(l-n)}}(p+l\psi^2)^{3/2}\right] +$$
$$+ [2l\sqrt{l(l-n)}\psi^2 + (\sqrt{l}-\sqrt{l-n})^2(p+l\psi^2)] \times$$
$$\times K^2_{1/3}\left[\frac{2}{3}\frac{\sqrt{l}-\sqrt{l-n}}{\sqrt{l(l-n)}}(p+l\psi^2)^{3/2}\right]\Big\}. \tag{3.9}$$

For the limiting angle of emission we obtain, as in classical theory, $\psi^2 = p/l$, or

$$\psi_0 = mc^2/E. \tag{3.10}$$

For $n \ll l$ the angle may be even greater.

Let us now proceed to the frequency of emission and express it through $y = \omega/\omega_{cr}$, where the critical frequency is

$$\omega_{cr} = \frac{c}{R}\left(\frac{E}{mc^2}\right)^3 \frac{3}{2+3(E/mc^2)^2(\hbar/mcR)} \tag{3.11}$$

---

[1] Quantum corrections found in [7] to the spectrum partially coincide with those which may be obtained from formula (3.9), but the most important correction term in intensity of the order of $n^{5/3}/l$ for $\theta = \pi/2$ and $n \ll (1-\beta^2)^{-3/2}$ (in our designations) is omitted there. Also not found in [7] is the correction to the spectrum integrated with respect to the angles and complete intensity.

and R is the radius of the orbit. Then we will find

$$I = \frac{9}{\pi^2}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4 \frac{1}{(2+z)^3}\int_0^{1+2/z} y^2\,dy\left\{\left[1+\frac{(2+z)^2}{[2+(1-y)z]^3}\right]\times\right.$$

$$\int_0^\infty K^2_{2/3}\left(\frac{y\,\mathrm{ch}^3 x}{2+(1-y)z}\right)\mathrm{ch}^5 x\,dx + 2\,\frac{2+z}{2+(1-y)z}\int_0^\infty K^2_{1/3}\left(\frac{y\,\mathrm{ch}^3 x}{2+(1-y)z}\right)\mathrm{ch}^3 x\,\mathrm{sh}^2 x\,dx +$$

$$\left. + \frac{z^2 y^2}{[2+(1-y)z]^2}\int_0^\infty K^2_{1/3}\left(\frac{y\,\mathrm{ch}^3 x}{2+(1-y)z}\right)\mathrm{ch}^5 x\,dx\right\}, \qquad (3.12)$$

where

$$z = 3\,(E/mc^2)^2\,(h/mcR). \qquad (3.13)$$

The spectral emission distribution (3.12) of the luminous electron (Figure 2) depends in quantum theory on a single parameter z, while in classical theory the form of the spectrum (as a function of ω/ω_{cr}) does not depend on the energy of the electron. The correctness of the passage to classical distribution to the limit for $z \to 0$ is easy to check by using the equation /27

$$\sqrt{3}\,y\int_y^\infty K_{5/3}(x)\,dx =$$

$$= \frac{3}{\pi}y^2\int_0^\infty dx\left\{K^2_{1/3}\left(\frac{y}{2}\,\mathrm{ch}^3 x\right)\mathrm{ch}^3 x\,\mathrm{sh}^2 x + K^2_{2/3}\left(\frac{y}{2}\,\mathrm{ch}^3 x\right)\mathrm{ch}^5 x\right\}, \qquad (3.14)$$

which is readily proved [3] by using Lerkh's Theorum.

For $z \to 0$, formula (3.11) for ω_{cr} turns into a classical expression, and for $E \to \infty$ the energy of the photon in the peak of the spectrum trends toward the energy of the electron.

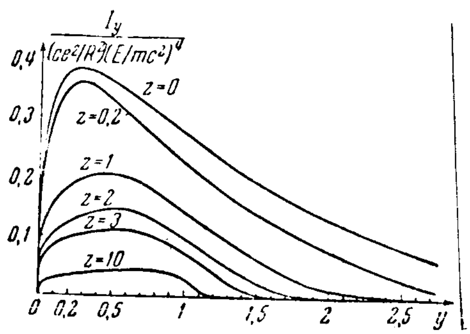

Fig. 2. Shape of Luminous Electron Emission Spectrum.

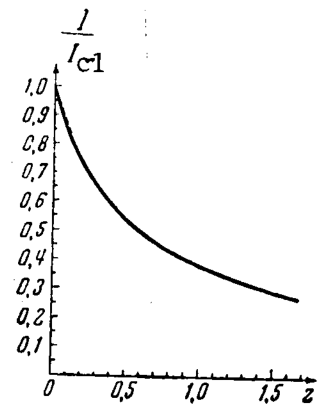

Fig. 3. Complete Intensity of Luminous Electron Emission.

4. Complete Intensity of Emission

Assuming $y\,\mathrm{ch}^3 x/[2+(1-y)z]=t$, $\mathrm{sh}\,x=\mathrm{tg}\,\theta$, will reduce (3.12) to the form

$$I=\frac{9}{\pi^2}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4\int_0^\infty t^2dt\int_0^{\pi/2}\frac{\cos^3\theta\,d\theta}{(1+tz\cos^3\theta)^4}\times$$

$$\times\{[1+(1+tz\cos^3\theta)^2]K^2_{2/3}(t)+[2\sin^2\theta\,(1+tz\cos^3\theta)+t^2z^2\cos^6\theta]\,K^2_{1/3}(t)\}. \quad (4.1)$$

The dependence of this integral on functions of z is illustrated in Figure 3. Using Formulas

$$\int_0^{\pi/2}\sin^s\theta\cos^p\theta\,d\theta=\frac{1}{2}\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{s+p}{2}+1\right)}, \quad (4.2)$$

$$\frac{1}{(1+x)^k}=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\,\Gamma(-s)\,x^s\frac{\Gamma(s+k)}{\Gamma(k)}, \quad (4.3)$$

$$\int_0^\infty K_\nu(x)\,K_\rho(x)\,x^{\mu-1}dx=$$

$$=\frac{2^{\mu-3}}{\Gamma(\mu)}\Gamma\left(\frac{\mu+\nu+\rho}{2}\right)\Gamma\left(\frac{\mu-\nu+\rho}{2}\right)\Gamma\left(\frac{\mu+\nu-\rho}{2}\right)\Gamma\left(\frac{\mu-\nu-\rho}{2}\right)\ (\mathrm{Re}\,\mu>|\mathrm{Re}\,(\nu\pm\rho)|), \quad (4.4)$$ /28

we will find (4.1) in the form of the integral

$$I=\frac{\sqrt{3}}{\pi^2}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\,z^{2s}\,\Gamma(-s)\,\Gamma\left(-s+\frac{1}{2}\right)\Gamma\left(s+\frac{7}{3}\right)\times$$

$$\times\left\{\frac{3}{2}\Gamma(s+1)\,\Gamma\left(s+\frac{3}{2}\right)\Gamma\left(s+\frac{2}{3}\right)+z^2\Gamma(s+2)\,\Gamma\left(s+\frac{5}{2}\right)\Gamma\left(s+\frac{5}{3}\right)\right\}. \quad (4.5)$$

Considering the reduction in the bands lying to the left of the contour, it is easy to find the expression for emission intensity in the form

$$I=\frac{2}{3}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4\Big\{z^{-4/3}\Gamma\left(\frac{2}{3}\right)\Big[{}_1F_2\left(\frac{7}{6};\ -\frac{2}{3},\ \frac{1}{6};\ \frac{1}{z^2}\right)+$$
$$+\frac{7}{9}\,{}_2F_3\left(\frac{5}{3},\ \frac{13}{6};\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{6};\ \frac{1}{z^2}\right)\Big]-z^{-2}\frac{9}{4}\Big[{}_2F_3\left(1,\ \frac{3}{2};\ \frac{1}{2},\ -\frac{1}{3},\ \frac{4}{3};\ \frac{1}{z^2}\right)+$$
$$+3\,{}_2F_3\left(3,\ \frac{5}{2};\ \frac{1}{2},\ \frac{2}{3},\ \frac{4}{3};\ \frac{1}{z^2}\right)\Big]+z^{-8/3}\frac{55}{3}\Gamma\left(\frac{1}{3}\right){}_2F_3\left(\frac{7}{3},\ \frac{17}{6};\ \frac{4}{3},\ \frac{5}{3},\ \frac{5}{6};\ \frac{1}{z^2}\right)+$$
$$+z^{-3}\frac{27\sqrt{3}}{5}\Big[{}_1F_2\left(2;\ \frac{1}{6},\ \frac{11}{6};\ \frac{1}{z^2}\right)-12\,{}_2F_3\left(3,\ \frac{5}{2};\ \frac{3}{2},\ \frac{7}{6},\ \frac{11}{6};\ \frac{1}{z^2}\right)\Big]-$$
$$-z^{-14/3}\frac{297}{20}\Gamma\left(\frac{1}{3}\right){}_1F_2\left(\frac{17}{6};\ \frac{8}{3},\ \frac{11}{6};\ \frac{1}{z^2}\right)\Big\}. \tag{4.6}$$

For $z \gg 1$ we obtain the expansion

$$I=\frac{2}{3}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4\left\{\frac{16}{9}\Gamma\left(\frac{2}{3}\right)z^{-4/3}-9z^{-2}+\frac{55}{3}\Gamma\left(\frac{1}{3}\right)z^{-8/3}-\frac{297}{5}\sqrt{3}z^{-3}+\cdots\right\}. \tag{4.7}$$

Shifting the contour in (4.5) to the right, we find

$$I=\frac{2}{3}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4\Big\{{}_4F_1\left(1,\ \frac{3}{2},\ \frac{2}{3},\ \frac{7}{3};\ \frac{1}{2};\ z^2\right)-\frac{55\sqrt{3}}{48}z\,{}_3F_0\left(2,\ \frac{7}{6},\ \frac{17}{6};\ z^2\right)+$$
$$+\frac{2}{3}z^2\,{}_4F_1\left(2,\ \frac{5}{2},\ \frac{5}{3},\ \frac{7}{3};\ \frac{1}{2};\ z^2\right)-\frac{385\sqrt{3}}{144}z^3\,{}_4F_1\left(3,\ \frac{5}{2},\ \frac{13}{6},\ \frac{17}{6};\ \frac{3}{2};\ z^2\right)\Big\}. \tag{4.8}$$

The first terms of the asymptotic expansion of this total for $z \ll 1$ have the form

$$I=\frac{2}{3}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4\left\{1-\frac{55\sqrt{3}}{48}z+\frac{16}{3}z^2-\frac{8855}{864}\sqrt{3}z^3+\ldots\right\}. \tag{4.9}$$

In particular, by limiting ourselves in (4.9) to the first two terms, we obtain the result

$$I=\frac{2}{3}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4\left\{1-\frac{55\sqrt{3}}{16}\frac{\hbar}{mcR}\left(\frac{E}{mc^2}\right)^2\right\}, \tag{4.10}$$

found also in [4]. In practical units we have

$$I/I_{cl} = 1 - 8{,}8 \cdot 10^{-6} E^2_{10^9\text{ev}} / R_{\text{M}}. \tag{4.11}$$

If the energies of the electrons vary in time according to the law $E = E_0 \sin(\pi t/2T)$ $(0 \leqslant t \leqslant T)$, we will have, instead of (4.10),

$$\bar{I} = \frac{1}{4}\frac{ce^2}{R^2}\left(\frac{E_0}{mc^2}\right)^4 \left\{1 - \frac{275\sqrt{3}}{96}\frac{h}{mcR}\left(\frac{E_0}{mc^2}\right)^2\right\}. \tag{4.12}$$ /29

The dependence of $\bar{I}$ on z is illustrated in Figure 3 by the broken line.

The first coefficient in (4.9) corresponds to those obtained earlier in [4] by a different method [2], having a significantly narrower range of applicability ($z << 1$).

From the formulas obtained it is clear that classical theory is valid for computing the emission intensity for the condition

$$E/mc^2 \ll (mcR/3h)^{1/3}, \tag{4.13}$$

also obtained in the above cited works on quantum theory of the luminous electron (see also [8.12]. Experimental studies [13-15] also pertain to greater limitations of the applicability of classical theory in computing emission intensity than condition (4.13).

Our theory,based on the assumption that $E/mc^2 >> 1$, would be valid if the peak of the emission spectrum falls to the last higher harmonics (the validity of the assumption $l' >> 1$ is violated). Assuming that the peak of (3.11) reaches the harmonic $n = l - 1$, we will find the condition of applicability of the proposed theory:

$$E/mc^2 \ll mcR/3h. \tag{4.14}$$

This energy is much greater than that of (4.13) for macroscopic values of the radius of the orbit.

5. Single-Photon Annihilation of Electrons and Positrons in a Magnetic Field

Let us examine [3] the transition of an electron from the state of positive energy in a magnetic field to the state of negative energy in the same field with the emission of a photon, making use of (3.1), (3.2) and assuming $\varepsilon' = -1$. We will select a coordinate system such that we will obtain $k_1 = k'_1$

and $k_3 = k'_3$. Then, averaging the probability of transition with respect to the electron and positron spins, and adding the values of the momentum of the photon, we will find the probability of annihilation per unit time

$$W = \frac{1}{L_x L_z} \frac{\pi}{2} \frac{e^2}{\hbar\varkappa} \sum_{s,\, s' = \pm 1} (|\alpha_x|^2 + |\alpha_z|^2), \tag{5.1}$$

whereupon $\kappa_1 = \kappa_3 = 0$ and $\kappa = K + K'$ because of the laws of conservation. This probability is not equal to zero, since part of the pair's momentum will incorporate the magnetic field. Since $\kappa_1 = 0$, the photon is emitted in a direction perpendicular to the line connecting the centers of rotation of the particles.

The multiplication of the probability of (5.1) by the number of electrons N with which a positron can be annihiliated is equal to the recriprocal of the lifetime of a positron in a given electron beam, where $n = N/L_x L_z$ is the density of the stream of electrons in the beam. We will recall that in the case of two-photon annihilation occuring in the absence of an external field, the recriprocal of the lifetime of a positron is proportional to the volumetric density of the electrons.

The effective cross-section of transformation cannot be defined here, since the incident waves cannot be regarded as planar at any distance. It is this very circumstance which explains the inability to regard a magnetic field in a given transformation as a perturbation of free travel, as may be done with the electrical field of the atomic nucleus. Our calculation is therefore analogous not to single-photon annihilation of positrons in atoms in the Bohr approximation, but to calculations dealing directly with the electrical field (a detailed list of references on the problems of the annihilation of electrons and positrons, as well as on the creation of pairs of these particles and the application of these phenomena to many problems in physics, is furnished in [3]). A determination of the effective cross-section is also impossible in the reverse transformation (creation of photon pairs in a magnetic field) since a uniform magnetic field is not a spatially restricted phenomenon. /30

Let us assume that $E \gg mc^2$ and $E' \gg mc^2$. Then, using (1.18), (1.20), (1.29), (1.30), and (1.31), together with (2.18) and (2.19), we obtain for the lifetime of a positron in a magnetic field

$$\frac{1}{\tau_H} = n \frac{cr_0}{6\pi} \frac{(mc^2)^5}{E^3 E'^3 (E + E')} \left[1 + \left(\frac{p_z}{mc}\right)^2\right] \times$$
$$\times \left\{\left[(E + E')^2 + \left(\frac{p_z}{mc}\right)^2 (E^2 + E'^2)\right] K^2_{1/3}\left[\frac{H_0}{3H} \frac{mc^2 (E + E')}{EE'} \left(1 + \left(\frac{p_z}{mc}\right)^2\right)^{3/2}\right] + \right.$$
$$\left. + (E^2 + E'^2) \left[1 + \left(\frac{p_z}{mc}\right)^2\right] K^2_{2/3}\left[\frac{H_0}{3H} \frac{mc^2 (E + E')}{EE'} \left(1 + \left(\frac{p_z}{mc}\right)^2\right)^{3/2}\right]\right\}. \tag{5.2}$$

In particular, for $E = E'$ and $p_z = 0$

$$\frac{1}{\tau_H} = ncr_0\left(\frac{H}{H_0}\right)^5 f\left(\frac{2}{3}\frac{H_0}{H}\frac{mc^2}{E}\right), \tag{5.3}$$

$$f(x) = \frac{81}{64\pi} x^5 \{2K^2_{2/3}(x) + K^2_{1/3}(x)\}. \tag{5.4}$$

The function $f(x)$ is shown graphically in Figure 4.

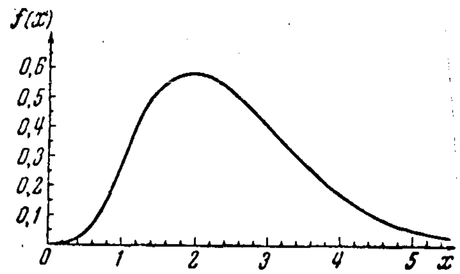

Fig. 4. Determination of the Probability of Single-Photon Annihilation of Electron and Positron in a Magnetic Field

For $x \gg 1$, $f(x) \approx \frac{243}{128} x^4 e^{-2x}$, and

$x \ll 1$, $f(x) \approx \frac{81\Gamma^2(2/3)\,2^{1/3}}{128\pi} x^{11/3}$.

The probability of annihilation reaches its maximum for

$$E_{max} = 0{,}33 mc^2 \frac{H_0}{H}. \tag{5.5}$$

In practical units

$$\frac{1}{\tau_{H\,sec}} = 3{,}52\cdot 10^{-26} n_{cm^{-3}} H^5_{10^9 Oe} f\left(\frac{15{,}9}{H_{10^9 Oe} E_{10^9 eV}}\right) \tag{5.6}$$

and

$$E_{max,\,10^9 eV} = \frac{7{,}8}{H_{10^9 Oe}}. \tag{5.7}$$

Here the energy of the particles is measured in billions of electron-volts, the magnetic field is measured in billions of Oersteds, density in cm^{-2}, /31
and time in seconds. The sharp decrease in probability with a decrease in energy and magnetic field justifies the use of our approximations, which are valid for high energies of the particles.

We will recall now that in the limiting case $H \gg H_0$ we have $q = \kappa^2/2\gamma \ll 1$ and for $l = l' \geq 1$ and $p_z \ll mc$, $|I(l', l)| = |I(l'-1, l-1)| = 1$, $|I(l', l-1)| = |I(l'-1, l)| \ll 1$, hence,

$$\frac{1}{\tau_H} = n\frac{\pi c r_0}{2}\sqrt{\frac{H_0}{2H}}. \tag{5.8}$$

To compare the probability of single-photon annihilation of electrons and positrons in a magnetic field with the probability of their two-photon annihilation, it is first necessary to find the ratio between densities n and ρ.

In a rather narrow beam of particles $\rho = n/2\pi R$, where $R = (v/c)(E/eH)$ and

$$r_0 \frac{\rho}{n} = \frac{1}{2\pi} \frac{e^2}{\hbar c} \sqrt{\frac{H}{2H_0}}. \tag{5.9}$$

If $H \ll H_0$, keeping in mind that for two-photon annihilation

$$\frac{1}{\tau_0} = \rho\pi c r_0^2 \frac{mc^2}{E} \ln \frac{E}{mc^2} \quad \text{for } \frac{v}{c} \sim 1,$$

we obtain for $E = E_{max}\ (H)$

$$\frac{\tau_{H\min}}{\tau_0} \sim \left(\frac{H_0}{H}\right)^2 \ln \frac{0{,}33 H_0}{H}. \tag{5.10}$$

Hence, the relative probability of single-photon annihilation here is low, but it increases as H increases. For $H \gg H_0$ and $v \ll c$, $1/\tau_0 = \rho\pi c r_0^2$ and from (5.8) and (5.9)

$$\frac{\tau_H}{\tau_0} \sim \frac{e^2}{\hbar c} \frac{\widetilde{H}}{H_0}. \tag{5.11}$$

In this case the relative probability of τ_0/τ_H decreases with the field.

When $H \gg H_0$ it is possible to make only a qualitative comparison, since the independence of τ_0 on H and the applicability of (5.9) in the last case are totally not apparent. These results do indicate, however, that when $H \sim H_0$ the ratio τ_H/τ_0 may be of the order of 1/137, i.e. single-photon annihilation will predominate (or will be at least of the same order of probability as two-photon annihilation). The possibility of the process of single-photon annihilation of electrons and positrons in a magnetic field has not yet been treated in literature.

6. Creation of Electron-Positron Pairs By Photons in a Magnetic Field

We will find [3] the probability of the phenomenon which is the antithesis of the above, i.e. the transformation of a photon to an electron-positron pair in a magnetic field--the unique nonlinear effect of the interaction of electromagnetic fields. By taking the complex conjugate matrix elements from the matrix elements of the preceding calculation and selecting a coordinate system such that $\kappa_1 = \kappa_3 = 0$, we will find $k_1 = k'_1$ and $k_3 = k'_3$, and for the probability of transition per unit time, averaged with respect to the polarizations of the photon, we obtain

$$W = \frac{e^2}{2\varkappa\hbar L_y} \int_{k_{1\min}}^{k_{1\max}} dk_1 \int_{-\infty}^{\infty} dk_3 \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \sum_{s,\, s'=\pm 1} (|\alpha_x|^2 + |\alpha_z|^2)\, \delta(K + K' - \varkappa). \tag{6.1}$$ /32

Keeping in mind that $|y_0| = |k_1|/\gamma \leqslant L_y/2$ and that the moduli of the matrix elements and the energy are independent of k_1, we find

$$W = \frac{1}{2}\frac{e^2}{\hbar c}\frac{mc^2}{\hbar}\frac{mc^2}{E}\frac{H}{H_0}\int_{-\infty}^{\infty} dk_3 \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{s,s'=\pm 1}\left(|\overline{\alpha}_x|^2 + |\overline{\alpha}_z|^2\right)\delta(K+K'-\varkappa). \tag{6.2}$$

The value $1/W$ is the lifetime of the photon in a magnetic field, and c/W is its path. If we consider a beam of photons with energy density $n(E)$ and with cross-section S, at distance d for time t there will be

$$N = St\int n(E)\,dE\left(1 - \exp\left\{-\frac{d}{c}\,W(E,H)\right\}\right) \tag{6.3}$$

transformations of photons into pairs. If $(d/c)\,W \ll 1$, the effect is proportional to the volume of the field.

It is easy to see that

$$\delta(K+K'-\varkappa) = \frac{KK'}{\varkappa k_3}\,\delta\left(k_3 \pm \sqrt{\frac{\varkappa^2}{4} - (l+l')\gamma - k_0^2 + \frac{(l-l')^2}{\varkappa^2}\gamma^2}\right). \tag{6.4}$$

From (6.4) it is clear that W is transformed to infinity each time $k_3 = 0$, i.e. when $\varkappa^\infty = \sqrt{k_0^2 + 2l\gamma} + \sqrt{k_0^2 + 2l'\gamma}$, where l and l' are integers.

If $l = l'$, for $E \gg mc^2$ (E is the energy of the photon) we have

$$\Delta\varkappa^\infty = \varkappa_{l+1}^\infty - \varkappa_l^\infty \approx \sqrt{2\gamma/l} \tag{6.5}$$

or

$$\frac{\Delta E^\infty}{mc^2} = 4\,\frac{H}{H_0}\frac{mc^2}{E}. \tag{6.6}$$

In practical units

$$\Delta E^\infty_{\text{ev}} = 2{,}22\cdot 10^{-8}\,\frac{H_{\text{Oe}}}{E_{\text{mev}}}. \tag{6.7}$$

We see that the singularities of probability for $H \ll H_0$ and $E \gg mc^2$ are very frequently distributed on the scale of photon energy. These singularities will inevitably be smoothed for the attainable range of the line with the distribution of $n(E)$. By using the law of conservation of energy we find that if we exclude l' instead of k_3, considering that this value changes constantly, we immediately find the probability with smooth singularities. Using (1.18), (1.20), (1.29), (1.30), (1.31), (2.18), and (2.19), expression (6.2) can be reduced to the form

$$W = \frac{e^2}{\hbar c}\frac{mc^2}{\hbar}\frac{H}{H_0}\varphi\left(\frac{4}{3}\frac{H_0}{H}\frac{mc^2}{E}\right), \tag{6.8}$$

where

$$\varphi(\varepsilon) = \frac{3}{2\pi^3}\varepsilon^2\int_0^\infty dx\int_0^\infty dy\,\{2\,\mathrm{ch}^2 y\,\mathrm{ch}^5 x\,K^2_{1/3}(\varepsilon\,\mathrm{ch}^2 y\,\mathrm{ch}^3 x) -$$
$$- \mathrm{sh}^2 x\,\mathrm{ch}^3 x\,K^2_{1/3}(\varepsilon\,\mathrm{ch}^2 y\,\mathrm{ch}^3 x) + (2\,\mathrm{ch}^2 y - 1)\,\mathrm{ch}^5 x\,K^2_{2/3}(\varepsilon\,\mathrm{ch}^2 y\,\mathrm{ch}^3 x)\}. \tag{6.9}$$

The approximation of high energies of particles and the photon is also used in this case, since when conditions $l \gg 1$, and $l' \gg 1$ are not satisfied, the probability is very low. The function $\phi(\varepsilon)$ is illustrated graphically in Figure 5, where /33

$$\varphi(\varepsilon) = \frac{5\Gamma(5/6)}{7\Gamma(7/6)\,2^{4/3}}\varepsilon^{1/3} \text{ for } \varepsilon \ll 1,$$

$$\varphi(\varepsilon) = \frac{3\sqrt{3}}{16\sqrt{2}}e^{-2\varepsilon} \text{ for } \varepsilon \gg 1.$$

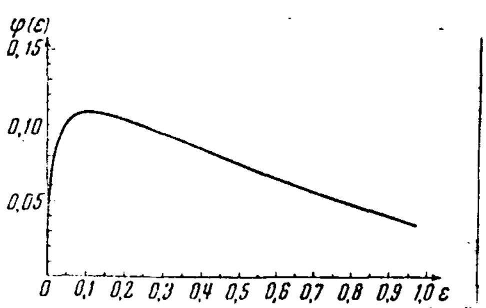

Fig. 5. Determining the Probability of Creation of a Pair by a Photon in a Magnetic Field.

The maximum of the probability of transformation is reached for

(6.10)

In practical units

$$W = 1{,}22\cdot 10^{11} H_{10^6\,\mathrm{Oe}}\,\varphi\left(\frac{31{,}8}{E_{10^6\,\mathrm{eV}} H_{10^6\,\mathrm{Oe}}}\right) \tag{6.11}$$

and

$$E_{\max 10^6\,\mathrm{ev}} = 274 / H_{10^6\,\mathrm{Oe}}. \tag{6.12}$$

The theory under discussion can indicate the cause of the great absorption of photons, even in weak fields, only if the field has sufficient magnitide, despite the fact that the high probability of transformation of a photon into a pair occurs only in very strong fields. In cosmic magnetic fields having a magnitude of the order of 10^{19} cm and a strength of the order of 10^{-5}--10^{-6} Oe., photons with energies higher than $4\cdot 10^{24}$ Oe should be completely smooth, whereupon the path of the photon with an energy of the order of 10^{25} ev has much smaller dimensions than the magnetic cloud. This effect will not have any influence on photons of the specified energy, or on the visible light of stars.

In fields having a strength of the order of 10^5 Oe the minimal path of the photon ($E \sim 10^{14}$ ev) is at most a few centimeters.

Only qualitative indications [8, 9][2] have, up to now, appeared on literature concerning the possibility of the transformation of a photon into a pair in a magnetic field.

7. Creation of Electron-Photon Pairs by Electrons In a Magnetic Field /34

To find the qualitative evaluation of the energy of electrons for which we should expect the intensive emission of pairs in a magnetic field by the electrons, we will find the energy of the electrons for which the peak of the emission spectrum of the luminous electron coincides with the maximum probability of the creation of a pair by a photon in a magnetic field. In accordance with (6.10) the energy of the photon here is

$$\varepsilon = 11 mc^2 \frac{H_0}{H} \tag{7.1}$$

or,

$$\varepsilon = 11 mc^2 \frac{mcR}{\hbar} \frac{mc^2}{E}. \tag{7.2}$$

if we express the magnetic field through the energy of the electron.

From (3.11) we find

$$\varepsilon = mc^2 \frac{\hbar}{mcR} \left(\frac{E}{mc^2}\right)^3 3 \left[2 + 3\left(\frac{E}{mc^2}\right)^2 \frac{\hbar}{mcR}\right]^{-1}. \tag{7.3}$$

[2] After the completion of this article we became aware of [16], in which the creation of pairs by photons in a magnetic field is discussed. The result of [16] is suggested for use only when the effect is very slight and contains the correct exponential factor. The other factors in [16] were found to be incorrect, since an incorrect expression was used for the density of final states and the approximation for the matrix elements was too rough (the diminution of probability with increasing values of $|l - l'|$ in our designations was incorrectly approximated.

By equating (7.2) and (7.3) we find

$$\frac{E}{mc^2} \sim 6\left(\frac{mcR}{3\hbar}\right)^{1/2}. \tag{7.4}$$

From (7.4) we see that the influence of the effect under consideration on energy losses of the particles in a magnetic field appears soon after the emission of photons begins to deviate from classical theory.

The author expresses his heartfelt graditude to professor A. A. Sokolov, who supervised the work on this theme.

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Translated for the National Aeronautics and Space Administration under contract No. NASw-1695 by Techtran Corporation, P.O. Box 729, Glen Burnie, Maryland 21061